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Simulation of 2D External Viscous Flows by Means of a Domain Decomposition Method

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Two-dimensional external viscous flows are numerically approximated by means of a domain decomposition technique which combines a vortex method and a finite differences method. The vortex method is used in the flow region which is dominated by convective effects, whereas the finite differences method is used in the flow region where viscous diffusion effects are dominant. Meanwhile, a new vortex method which is suitable for approximating first-order linear hyperbolic problems supplemented with Dirichlet boundary conditions is presented. Comparison with numerical and experimental data show that the method is well adapted for calculating two-dimensional external flows at moderate Reynolds numbers. © 1993 Academic Press, Inc.

1. INTRODUCTION

Consider a fluid domain Ω in $\mathbb{R}^2$. Denote by L the macroscopic length scale of Ω, and denote by u_∞ the velocity scale of the flow. Assume the fluid motion is incompressible and satisfies Navier–Stokes equations. Denote the fluid's kinematic viscosity by ν and the Reynolds number by $\mathrm{Re} := u_\infty L/\nu$. It is well known that the laminar dissipation scale of the fluid motion is of order $L/\mathrm{Re}^{1/2}$. Hence, in order to resolve the dissipation scale, every numerical method that is based on an Eulerian description of the flow, that is with a fixed, nonadaptive grid, requires a number of grid points that is of $\mathcal{O}(\mathrm{Re})$. As a consequence, capacities of computers are rapidly exceeded as the Reynolds number increases. Hence, alternative approaches must be found for moderate and high Reynolds numbers.

For external flows at moderate and high Reynolds numbers, vorticity concentrates in boundary layers and wakes whose characteristic length scale is of order $L/\mathrm{Re}^{1/2}$. Vortex methods, introduced by Chorin [2], are based on the Lagrangian formulation of Navier–Stokes equations and are known for tracing vorticity. Therefore, this class of method naturally concentrates discretization points in the regions of interest. As a result, we may infer that vortex methods would need only $\mathcal{O}(\mathrm{Re}^{1/2})$ discretization points to capture the dissipation scale, so they would be better candidates for describing external flows for which convection dominates viscous diffusion than the methods that are based on the Eulerian formulation of Navier–Stokes equations. Unfortunately, the major drawback of vortex methods is that boundary conditions cannot be easily prescribed; besides, they may be inaccurate in the flow regions where viscous effects dominate convective effects. The reader is referred to Chorin [2], Leonard [10], and Raviart [13] for reviews on vortex methods.

Conversely, methods that are based on the Eulerian formulation of Navier–Stokes equations (that is to say, finite differences methods, finite element methods, and the like) are well adapted for taking into account various forms of boundary conditions. Furthermore, they are highly efficient when applied on bounded domains and are particularly efficient in the regions where the elliptic nature of the flow is dominant (i.e., viscous effects dominate convective effects). Hence, this class of methods should be the natural counterpart of vortex methods.

The purpose of the present paper is to report on a domain decomposition technique that is based on the ideas above for calculating two-dimensional external flows at moderate Reynolds numbers. The flow domain is decomposed into two subdomains. Near the solid boundary, where viscous effects are dominant, the fluid motion is approximated by means of a finite differences scheme, far from the solid boundary, where convection is dominant, the flow is modelled by a vortex method (see Cottet [3] and Huberson *et al.* [8]), Shen [14]). The outline of the paper is as follows: In Section 2 we present the domain decomposition method, special emphasis is put on the time discretization and the coupling conditions. A theoretical presentation of a particle method that can take into account Dirichlet data is done in Section 3. Its numerical implementation is presented in Section 4. The finite differences technique which is used in the inner subdomain is reviewed in Section 5. Comparisons with experimental and other numerical results are reported in the sixth section.

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2. THE DOMAIN DECOMPOSITION METHOD

2.1. *Introduction*

The present approach is based on the domain decomposition method without overlapping [8, 9, 12, 14]. For a review on the domain decomposition techniques, the reader is referred to the proceedings of the "International Symposiums on Domain Decomposition Methods for Partial Differential Equations" which have been held yearly since 1987. Introductions of some domain decomposition methods for viscous flow problems can be found in Quarteroni [12] and Le Tallec [9]. Recall that the advantages of the domain decomposition technique are twofold: well-adapted numerical approximations can be used in each subdomain and each subdomain's solution can be calculated in parallel.

2.2. *Formulation of the Problem*

Let $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ be a direct, orthogonal, normed basis of $\mathbb{R}^3$. $\mathbb{R}^2$ is embedded in $\mathbb{R}^3$ so that $(\mathbf{i}, \mathbf{j})$ is a direct, orthogonal, normed basis of $\mathbb{R}^2$. Consider Oxy a cartesian coordinate system associated with the basis $(\mathbf{i}, \mathbf{j})$. Denote by Ω the unbounded fluid domain; assume that the boundary of Ω is homeomorphic to a circle, i.e., Ω is one-connected (a domain is said to be p-connected if its fundamental group has p generators, i.e., in 2D the domain has p holes). Denote by B_1 the boundary of Ω; the complement of Ω is referred to as the obstacle, see Fig. 1 for details of the notations. Assume that at infinity the fluid is moving with the velocity $\mathbf{u}_\infty(t)$ which is parallel to the Ox axis and which magnitude may vary in time. The obstacle is assumed to be at rest in the coordinate system Oxy.

Define (Ω_0, Ω_1) a partition of Ω so that the obstacle has no boundary in common with Ω_0 and Ω_1 is homeomorphic to a ring (see Fig. 1). Denote by Γ_1 the interface between Ω_0 and Ω_1. Henceforth, we solve the Navier–Stokes problem in Ω_0 and in Ω_1 in parallel.

In the domain Ω_0 the problem is formulated in terms of velocity and vorticity $(\mathbf{u}, \omega)$. If the whole fluid domain was to be considered, the system of equations to be solved would read

$$\begin{aligned}
\frac{\partial \omega}{\partial t} + \nabla \cdot (\omega \mathbf{u}) &= \nu \nabla^2 \omega \\
\nabla \times \mathbf{u} &= \omega \mathbf{k} \\
\nabla \cdot \mathbf{u} &= 0 \\
\mathbf{u} &= 0 \qquad \text{on } B_1 \\
\int_{B_1} \frac{\partial \omega}{\partial n}\, dl &= 0 \\
\mathbf{u} &\xrightarrow{\infty} \mathbf{u}_\infty(t).
\end{aligned} \tag{2.1}$$

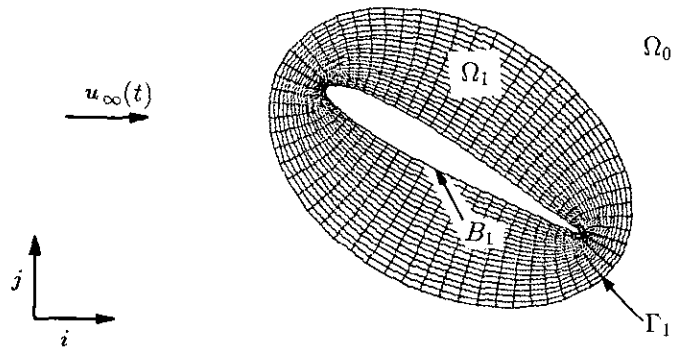

FIG. 1. Definition of notations.

In the following, the $(\mathbf{u}, \omega)$ formulation is restricted to the domain Ω_0.

In the domain Ω_1 the problem is formulated in terms of stream function and vorticity (ψ, ω). Recall that incompressibility, together with the condition $\int_{B_1} \mathbf{u} \cdot \mathbf{n}\, dl = 0$, implies that there is a stream function ψ so that $\mathbf{u} = \nabla \times (\psi \mathbf{k})$, where $\mathbf{n}$ is the outward unit normal to the obstacle's boundary. The (ψ, ω) formulation of Navier–Stokes equations in the whole fluid domain, Ω, reads:

$$\begin{aligned}
\frac{\partial \omega}{\partial t} + \nabla \cdot (\omega \nabla \times (\psi \mathbf{k})) &= \nu \nabla^2 \omega \\
\nabla^2 \psi &= -\omega \\
\psi &= \psi_{B_1} \qquad \text{on } B_1 \\
\frac{\partial \psi}{\partial n} &= 0 \qquad \text{on } B_1 \\
\int_{B_1} \frac{\partial \omega}{\partial n}\, dl &= 0 \\
\psi &= u_\infty(t)\, \mathbf{x} \cdot \mathbf{j} + o(1) \qquad \text{as} \quad |x| \to \infty.
\end{aligned} \tag{2.2}$$

The condition $\int_{B_1} \partial\omega/\partial n\, dl = 0$ is necessary to ensure that the pressure is a uniform function. Indeed, it is this very condition that imposes the value of the stream function, ψ_{B_1}, on the boundary B_1. Henceforth, we restrict the (ψ, ω) formulation to the bounded domain Ω_1.

Let $T > 0$ and N be an integer; the solution is sought in the time interval $[0, T]$. We set $\delta t = T/N$ and $t_k = k\, \delta t$ for $0 \leqslant k \leqslant N$. Assume that $(\mathbf{u}_0^l, \omega_0^l)$ and (ψ_1^l, ω_1^l) are known approximations of $(\mathbf{u}, \omega)$ and (ψ, ω) in domains Ω_0 and Ω_1 on time intervals (t_{l-1}, t_l) for $1 \leqslant l \leqslant k \leqslant N-1$; we wish to define approximations of $(\mathbf{u}, \omega)$ and (ψ, ω) in domains Ω_0 and Ω_1 on the time interval (t_k, t_{k+1}).

2.3. *Solution in Ω_0*

In the following, we restrict the $(\mathbf{u}, \omega)$ formulation to the domain Ω_0, and the partition of Ω is assumed to be chosen

so that in Ω_0 viscous diffusion is dominated by convection. Furthermore, in Ω_0 the Navier–Stokes problem is solved by means of a particle method. This kind of method roughly consists of advecting and diffusing information along the flow's characteristics. One advantage of such a procedure is that stability is not greatly spoiled when the velocity field is discretized explicitly in time. As a result, the convection-diffusion equation governing ω_0^{k+1} is hereafter approximated by

$$\frac{\partial \omega_0^{k+1}}{\partial t} + \nabla \cdot (\omega_0^{k+1} \tilde{\mathbf{u}}^{k+1}) = \nu \nabla^2 \omega_0^{k+1}, \tag{2.3}$$

where $\tilde{\mathbf{u}}^{k+1} = \mathbf{u}^k$ for a first-order approximation in time or $\tilde{\mathbf{u}}^{k+1} = (3\mathbf{u}^k - \mathbf{u}^{k-1})/2$ for a second-order approximation in time. For each time step the velocity field is given by Biot and Savart's law,

$$\mathbf{u}^k(\mathbf{x}) = \mathbf{u}_\infty(t_k) + \int_{\Omega_0 \cup \Omega_1} \omega^k(\mathbf{y}) \nabla G(\mathbf{y} - \mathbf{x}) \times \mathbf{k} \, dv, \tag{2.4}$$

where $G(\mathbf{y} - \mathbf{x}) := 1/2\pi \log(|\mathbf{y} - \mathbf{x}|)$ is Green's function of Laplace's operator in $\mathbb{R}^2$. It is shown in Section 4 how Eqs. (2.3) and (2.4) can be discretized by means of a particle method. Note that, since velocity is explicit with respect to time and viscous diffusion is small, Eq. (2.3) may be viewed in a first approximation as a linear hyperbolic equation. Transmission of information between subdomains Ω_0 and Ω_1 is hereafter based on this remark.

Wellposedness of the problem on the vorticity requires some transmission condition between ω_0 and ω_1 to be enforced across the interface Γ_1. The condition in question is dictated by the fact that, since viscosity is small, ω_0^{k+1} is the solution to a first-order hyperbolic problem in a first approximation. Recall that for this kind of PDE, the boundary value of ω_0^{k+1} is imposed only on the part of Γ_1 where the flow enters Ω_0 (upwind boundary condition); on the rest of the boundary ω_0^{k+1} is computed by solving the partial differential equation (2.3) on Ω_0 (downwind condition). Therefore, a boundary condition must be specified on the subset of Γ_1, where the velocity, $\mathbf{u}^k$, is such that $\mathbf{u}^k \cdot \mathbf{n}_0 < 0$, where $\mathbf{n}_0$ is the unit outward normal to the boundary of Ω_0. On the subset of Γ_1, where $\mathbf{u}^k \cdot \mathbf{n}_0 > 0$, the value of ω_0^{k+1} is computed by solving (2.3). For further details on the domain decomposition technique for hyperbolic problems, the reader is referred to Le Tallec [9] and Quarteroni [12].

In view of the remarks above, the transmission condition that is needed reads

$$\text{On } \Gamma_1 : \omega_0^{k+1} = \omega_1^k(\mathbf{x} - \delta t \, \mathbf{u}^k) + \mathcal{O}(\nu \, \delta t), \qquad \text{if} \quad \mathbf{u}^k \cdot \mathbf{n}_0 < 0. \tag{2.5}$$

This transmission condition is only first-order accurate in time. A transmission condition that is second-order accurate in time can be obtained by looking for an approximate parabolic decomposition of the advection-diffusion operator as shown in [1].

In conclusion, the problem to be solved in Ω_0 reduces to solving the almost hyperbolic equation (2.3) supplemented with the transmission condition (2.5), the velocity field being given by Biot and Savart's law (2.4).

2.4. *Solution in Ω_1*

In the inner domain Ω_1, the system of partial differential equations to be solved reads

$$\frac{\partial \omega_1^{k+1}}{\partial t} + \nabla \cdot (\omega_1^{k+1} \nabla \times (\psi_1^{k+1} \mathbf{k})) = \nu \nabla^2 \omega_1^{k+1}$$
$$\nabla^2 \psi_1^{k+1} = -\omega_1^{k+1}. \tag{2.6}$$

This system is supplemented with the following boundary conditions on B_1:

$$\psi_1^{k+1} = \psi_{B_1}^{k+1}, \qquad \frac{\partial \psi_1^{k+1}}{\partial n} = 0, \qquad \int_{B_1} \frac{\partial \omega_1^{k+1}}{\partial n} \, dl = 0. \tag{2.7}$$

Furthermore, the (ψ, ω) formulation of the Navier–Stokes problem being restricted to the bounded domain Ω_1, the conditions at infinity on ψ, which appear in (2.2), must be replaced by transmission conditions on ψ and ω.

Since whatever the flow nature (convective, diffusive, or both) the stream function is always the solution to an elliptic problem ($\nabla^2 \psi = -\omega$), a transmission condition is required on the whole boundary Γ_1. This transmission condition may be of Dirichlet, Neumann, or Robin type. A condition of Dirichlet type seemed natural to us; that is, we chose to impose ψ on Γ_1. One way to compute ψ_1^{k+1} on Γ_1 consists of using Green's third identity; in other words, for all $\mathbf{x}$ in Γ_1 the boundary value $\psi_1^{k+1}(\mathbf{x})$ is given by

$$\psi_1^{k+1}(\mathbf{x}) = u_0(t_{k+1}) \, \mathbf{x} \cdot \mathbf{j} - \int_{\Omega_0 \cup \Omega_1} G(\mathbf{y} - \mathbf{x}) \, \omega^k(\mathbf{y}) \, dv$$
$$- \psi_{B_1}^{k+1} \int_{B_1} \frac{\partial G(\mathbf{y} - \mathbf{x})}{\partial n} \, dl. \tag{2.8}$$

This transmission condition is first-order accurate in time. Note also that it is of integral type; as a result, it transmits the whole spectrum of information from Ω_0 to Ω_1 at once, whereas classical Dirichlet–Neumann coupling conditions poorly transmit low frequencies (see Quarteroni [12] for further details on the Dirichlet–Neumann technique). It is shown in Section 5 how (2.8) can be discretized by means of a particle method.

Let us now determine some transmission condition for the vorticity. From the analysis of the problem in Ω_0, it has been shown that ω_1^{k+1} is necessarily imposed on the subset

of Γ_1, where the flow enters Ω_1. Hence, the transmission condition on the vorticity ω_1^{k+1} must be computed by solving (2.3) approximately; in other words, ω_1^{k+1} is given by a first-order accurate upwind integration,

$$\text{On } \Gamma_1 : \omega_1^{k+1}(\mathbf{x}) = \omega_0^k(\mathbf{x} - \delta t\, \mathbf{u}^k) + \mathcal{O}(\nu\, \delta t), \quad \text{if} \quad \mathbf{u}^k \cdot \mathbf{n}_0 > 0. \tag{2.9}$$

Furthermore, since we want to approximate ω_1^{k+1} in Ω_1 by means of a numerical technique which is well adapted to elliptic problems, namely finite differences or finite elements, we also need to specify a boundary condition on ω_1 on the part of Γ_1, where the flow goes out of Ω_1. The boundary condition in question can by no means be obtained from Ω_0, for in Ω_0 the problem is hyperbolic and for such a flow information cannot go upwind. Consequently, the piece of information that is required must be looked for in the upwind direction in Ω_1. The natural way of doing so still consists of assuming that in the vicinity of Γ_1 the flow is dominated by convective effects; hence, in the vicinity of Γ_1, ω_1^{k+1} is locally solution to a first-order hyperbolic equation of type (2.3), and ω_1^{k+1} is given once more by an upwind integration of the type

$$\omega_1^{k+1}(\mathbf{x}) = \omega_1^k(\mathbf{x} - \delta u\, \mathbf{u}^k) + \mathcal{O}(\nu\, \delta t). \tag{2.10}$$

Note that this condition is not a transmission condition since it does not use information coming from the domain Ω_0. The condition (2.10) is merely required for computing convenience.

In conclusion, in domain Ω_1 the boundary condition on ω_1^{k+1} reads

$$\text{For all } \mathbf{x} \in \Gamma_1, \qquad \omega_1^{k+1}(\mathbf{x}) = \omega_i^k(\mathbf{x} - \delta t\, \mathbf{u}^k) + \mathcal{O}(\nu\, \delta t), \tag{2.11}$$

where $\omega_i^k = \omega_0^k$ if $\mathbf{u}^k \cdot \mathbf{n}_1 < 0$ (transmission condition) and $\omega_i^k = \omega_1^k$ if $\mathbf{u}^k \cdot \mathbf{n}_1 \geqslant 0$ (boundary condition), and $\mathbf{n}_1$ denotes the unit outward normal to the boundary of Ω_1. Note that $\mathbf{n}_0 = -\mathbf{n}_1$.

3. PARTICLE APPROXIMATION IN Ω_0

3.1. *Statement of the Problem*

Recall that in domain Ω_0, at each time step (t_k, t_{k+1}), the vorticity field is solution to the almost-hyperbolic equation (2.3), where the velocity field is explicitly given by formula (2.4). Actually the model problem to be solved can be cast into the form

$$\begin{aligned} &\frac{\partial \omega}{\partial t} + \nabla \cdot (u\mathbf{u}) = f(\mathbf{x}, t) \\ &\omega(\mathbf{x}, 0) = 0 \qquad \text{for} \quad \mathbf{x} \in \Omega_0 \\ &\omega(\mathbf{x}, t) = \omega_1(\mathbf{x}, t) \qquad \text{for} \quad \mathbf{x} \in \Gamma_1 \text{ such that } \mathbf{u} \cdot \mathbf{n}_0 < 0, \end{aligned} \tag{3.1}$$

where the velocity field is known at all times in the domain Ω_0. As mentioned in the introduction, we wish to solve (3.1) by means of a particle method. The difficulty at this point is that classical particle methods cannot account for Dirichltet conditions (see Raviart [13] for an analysis of particle methods). In order to overcome this difficulty, the classical technique of particle methods [13] has been modified and a somewhat new approach is proposed in the sequel.

Before solving problem (3.1), we need to discuss the representation of functions in Lagrange coordinates, together with their approximation by means of measures. For this purpose we introduce some notations.

Let $\partial\Omega_T = \Gamma_1 \times [0, T]$ and denote by $\partial\Omega_T^-$ and $\partial\Omega_T^+$ the subsets of $\partial\Omega_T$ for which $\mathbf{u} \cdot \mathbf{n}_0 < 0$ and $\mathbf{u} \cdot \mathbf{n}_0 > 0$, respectively. Consider $\Omega_0 \times]0, T[$ as an open cylinder in $\mathbb{R}^3$ and extend the definition of the normal $\tilde{\mathbf{n}}_0$ to the boundary of this cylinder. Define the generalized flow field $\tilde{\mathbf{u}}(\mathbf{x}, t) = (\mathbf{u}(\mathbf{x}, t), 1)$ in $\Omega_0 \times]0, T[$ and set $\Lambda^- = \partial\Omega_T^- \cup (\Omega_0 \times \{0\})$ and $\Lambda^+ = \partial\Omega_T^+ \cup (\Omega_0 \times \{T\})$. The subset Λ^- and Λ^+ are such that $\tilde{\mathbf{u}} \cdot \mathbf{n}_0 < 0$ and $\tilde{\mathbf{u}} \cdot \tilde{\mathbf{n}}_0 > 0$, respectively; that is, the generalized flow field $\tilde{\mathbf{u}}$ enters the cylinder $\Omega_0 \times]0, T[$ through Λ^- and exits it through Λ^+.

Since Ω_0 is regular (say Lipschitzian) and $\mathbf{u}(\mathbf{x}, t)$ is smooth (say continuous with respect to $(\mathbf{x}, t)$ and Lipschitzian with respect to $\mathbf{x}$), for every couple $(\mathbf{x}, t)$ in $\Omega_0 \times]0, T[$ there are two couples (ξ^-, τ^-) and (ξ^+, τ^+) in Λ^- and Λ^+, respectively, so that the characteristic curves of the generalized flow $\tilde{\mathbf{u}}$ which passes at $(\mathbf{x}, t)$ crosses Λ^- and Λ^+ at (ξ^-, τ^-) and (ξ^+, τ^+), respectively. Denote by $\psi : \Lambda^- \times]0, T[\to \Omega_0$, the mapping, so that $\psi(\xi^-, \tau^-, t) = x$. The mapping ψ is such that

$$\begin{aligned} &\frac{d\psi}{dt} = \mathbf{u}(\mathbf{x}, t), \qquad \tau^- \leqslant t \leqslant \tau^+ \\ &\psi(\xi^-, \tau^-, \tau^-) = \xi^-. \end{aligned} \tag{3.2}$$

Consider the time as a fixed parameter and define the mapping $\psi_t : (\xi^-, \tau^-) \mapsto \mathbf{x} = \psi(\xi^-, \tau^-, t)$. Let $J(\xi^-, \tau^-, t)$ be the jacobian determinant of the mapping ψ_t. A classical calculation yields

$$\frac{\partial J}{\partial t}(\xi^-, \tau^-, t) = J(\xi^-, \tau^-, t)\, \nabla \cdot \mathbf{u}(\mathbf{x}, t). \tag{3.3}$$

Since in the present case the velocity field is divergence free, J is known at all times, $\tau^- \leqslant t \leqslant \tau^+$, if it can be evaluated at the initial time τ^-. A simple calculation shows that

$$J(\xi^-, 0, 0) = 1, \qquad \text{if} \quad (\xi^-, \tau^-) \in \Omega_0 \times \{0\} \tag{3.4}$$

$$J(\xi^-, \tau^-, \tau^-) = -\mathbf{u} \cdot \mathbf{n}_0, \quad \text{if} \quad (\xi^-, \tau^-) \in \partial\Omega_T^-. \tag{3.5}$$

3.2. *Particle Approximation of Functions*

In the following we look for measure approximations of smooth functions $g : \Omega_0 \times]0, T[\to \mathbb{R}$ whose geometric sup-

port in Ω_0 at each time t is such that $\psi_t^{-1}(\mathrm{supp}(g(\cdot, t))) \subset \partial\Omega_T^-$; this kind of function is zero at time $t=0$. Note that the vorticity distribution which is solution to our model problem (3.1) belongs to the class of functions above.

Assume there exists a quadrature formula over $\partial\Omega_T^-$ so that for a certain class of functions $\phi(\mathbf{x}, t)$ we have

$$\int_{\partial\Omega_T^-} \phi(\xi^-, \tau^-)\, d\xi^-\, d\tau^- = \sum_{j\in J} \alpha_j \phi(\xi_j^-, \tau_j^-), \quad (3.6)$$

where $(\xi_j^-, \tau_j^-)_{j\in J}$ are quadrature points and $(\alpha_j)_{j\in J}$ are positive weights. Then, for every smooth function $\phi(\mathbf{x})$ with compact support in Ω_0 we have

$$\begin{aligned}\int_{\Omega_0} g(\mathbf{x}, t)\,\phi(\mathbf{x})\, d\mathbf{x} = \int_{\partial\Omega_T^-} & g(\psi_t(\xi^-, \tau^-), t) \\ & \times \phi(\psi_t(\xi^-, \tau^-))\, J(\xi^-, \tau^-, t) \\ & \times \chi_t(\xi^-, \tau^-)\, d\xi^-\, d\tau^-, \end{aligned} \quad (3.7)$$

where χ_t is a step function so that

$$\chi_t(\xi^-, \tau^-) = 0 \quad \text{if} \quad (\xi^-, \tau^-) \notin \mathrm{supp}(g(\cdot, t)) \quad (3.8)$$

$$\chi_t(\xi^-, \tau^-) = 1 \quad \text{if} \quad (\xi^-, \tau^-) \in \mathrm{supp}(g(\cdot, t)). \quad (3.9)$$

Hence, setting $\mathbf{x}_j(t) = \psi(\xi_j^-, \tau_j^-, t)$, $a_j(t) = \alpha_j J(\xi_j^-, \tau_j^-, t)$, and $\chi_j(t) = \chi_t(\xi_j^-, \tau_j^-)$, and by using the quadrature formula (3.6), Eq. (3.7) reduces to

$$\int_{\Omega_0} g(\mathbf{x}, t)\,\phi(\mathbf{x})\, d\mathbf{x} \approx \sum_{j\in J} a_j(t)\,\chi_j(t)\, g(\mathbf{x}_j(t), t)\,\phi(\mathbf{x}_j(t)). \quad (3.10)$$

That is, we have approximated the function g by the measure,

$$\Pi^h g(\mathbf{x}, t) = \sum_{j\in J} a_j(t)\,\chi_j(t)\, g(\mathbf{x}_j(t), t)\,\delta(\mathbf{x} - \mathbf{x}_j(t)), \quad (3.11)$$

where δ is the Dirac measure. Indeed, (3.11) is an approximation, in terms of measures, of the representation of g in Lagrange coordinates. Note that the so-called particle representation above is similar to the classical particle representation of functions whose support is not confined to a subdomain of $\mathbb{R}^2$ (see Raviart [13]). Actually, the present representation differs from the classical one in the use of the clipping functions $\chi_j(t)$. For a given index j, the function $\chi_j(t)$ means that the jth particle appears in Ω_0 at time τ_j^- at the location ξ_j^-, and it disappears at time τ_j^+ at the location ξ_j^+.

3.3. *Particle Solution of a Convection Equation*

In order to apply the above considerations to problem (3.1), we need to evaluate the convective derivative, in the sense of distributions, of measures of type (3.11). A simple calculation shows that, if no particular care is taken, the time derivative of $\Pi^h g$ yields terms which are proportional to $\delta(t-\tau_j^-)$ and $\delta(t-\tau_j^+)$. These terms are generated by the clipping function; as a result, in order to prevent the occurrence of such terms, the clipping function has to be regularized. In other words, a suitable particle approximation of the solution to (3.1) should be of the form,

$$\Pi_\varepsilon^h \omega(\mathbf{x}, t) = \sum_{j\in J} a_j \chi_{j\varepsilon}(t)\,\omega_j(t)\,\delta(\mathbf{x} - \mathbf{x}_j(t)), \quad (3.12)$$

where $\chi_{j\varepsilon}(t)$ are the regularized approximations of the clipping functions and ε is a regularization parameter. The fact that the clipping functions must be regularized means that the particles are not allowed to appear or disappear all of a sudden; their strength must grow and decrease smoothly as they enter the domain or leave it. Now, it can be verified that, in the sense of distributions on $\Omega_0 \times]0, T[$, we have

$$\frac{\partial \Pi_\varepsilon^h \omega}{\partial t} + \nabla\cdot(\Pi_\varepsilon^h \omega \mathbf{u}) = \sum_{j\in J} \frac{d}{dt}(a_j \chi_{j\varepsilon}(t)\,\omega_j(t))\,\delta(\mathbf{x} - \mathbf{x}_j(t)). \quad (3.13)$$

Note that, since the flow field is incompressible, the coefficients a_j are time independent and are given by

$$a_j = -\alpha_j \mathbf{u}(\xi_j^-, \tau_j^-)\cdot \mathbf{n}_0(\xi_j^-). \quad (3.14)$$

Henceforth, coefficients a_j are referred to as the volume of the jth particle.

If we now define as above a regularized particle approximation, $\Pi_\varepsilon^h f$, of the source term of problem (3.1), it is then a simple matter to verify that the particle approximation of problem (3.1) consists of finding the functions $t \mapsto \omega_j(t)$ which are solutions to

$$\begin{aligned} &\frac{d}{dt}[\chi_{j\varepsilon}(t)\,\omega_j(t)] = \chi_{j\varepsilon}(t) f(\mathbf{x}_j(t), t), \qquad \tau_j^- \leqslant t \leqslant \tau_j^+ \\ &\omega_j(\tau_j^-) = \omega_1(\xi_j^-, \tau_j^-). \end{aligned} \quad (3.15)$$

Furthermore, the particle positions $\mathbf{x}_j(t)$ are solutions to the differential equations,

$$\begin{aligned} &\frac{d}{dt}\mathbf{x}_j(t) = \mathbf{u}(\mathbf{x}_j(t), t), \qquad \tau_j^- \leqslant t \leqslant \tau_j^+ \\ &\mathbf{x}_j(\tau_j^-) = \xi_j^-. \end{aligned} \quad (3.16)$$

A theoretical analysis of the convergence of the present particle method, although highly desirable, has not yet been carried out. Indeed, such an analysis would result in optimum choices for the regularization of the clipping functions. Up to now the regularized clipping functions have been chosen by a trial and error procedure (see Sections 4.1 and 4.3 below). In the next section, we present a numerical implementation of the algorithm above, where the source term has been replaced by viscous diffusion.

4. NUMERICAL IMPLEMENTATION OF THE VORTEX METHOD

In the sequel we present a discrete version in time of the algorithm above (3.12), (3.14), (3.15), and (3.16). Assume that over the time interval $[0, t_k]$ production and absorption of particles yielded a set J_k of particles. The problem now consists of calculating approximations of the positions and strengths of the new set of particles J_{k+1} on the time interval (t_k, t_{k+1}). The algorithm above may be summarized as follows:

- Smoothly create new particles.
- Convect and diffuse the particles of J_k by means of any proper vortex method.
- Smoothly absorb the outgoing particles.

4.1. *Generation of New Particles*

We now discuss how new particles are smoothly generated. Recall that the new particles which enter the cylinder $\Omega_0 \times]0, T[$ are generated on the hypersurface $\partial\Omega_T^-$ at the points $(\xi_j^-, \tau_j^-)_{j\in J}$. Since the velocity field is not known in advance, the exact geometry of $\partial\Omega_T^-$, along with the quadrature points and weights of rule (3.6), cannot be defined in advance. As a consequence, $\partial\Omega_T^-$ and the quadrature points of rule (3.6) must be constructed progressively for each time interval (t_k, t_{k+1}). For this purpose, let us partition Γ_1 into a series of I segments of length δl_i. The middle of the ith segment is denoted by ξ_i. Let I_k^- be the subset of I which consists of the points ξ_i^- so that

$$\mathbf{u}(\xi_i^-, t_k)\cdot\mathbf{n}_0(\xi_i^-) < -\varepsilon, \tag{4.1}$$

where ε is a small positive parameter which prevents particles of negligible strength to be generated. Henceforth, the quadrature rule associated with the hypersurface $\partial\Omega_T^-$ is defined by the set of quadratures points $(\xi_i^-, \tau_k^-)_{i\in I_k^-, k=0,\dots,N-1}$ and weights $(\delta l_i\, \delta t)_{i\in I_k^-}$, where $\tau_k^- = (t_k + t_{k+1})/2$.

Henceforth, the points $(\xi_i)_{i\in I}$ are called spots, and with each spot are associated two flags. One flag indicates whether the spot is active at the present time step, it is referred to as the activity flag. The other flag indicates whether the spot is in the process of growing a particle, it is referred to as the history flag; the particle which is in the process of growing is referred to as a newborn particle which is still linked to its generating spot. At time t_0 all the spots' flags are lowered.

The algorithm for smoothly generating particles during the time interval (t_k, t_{k+1}) may be put into the following form:

- For each spot ξ_i, compute the normal component of the velocity $u_n = \mathbf{u}(\xi_i, t_k)\cdot\mathbf{n}_0(\xi_i)$. If $u_n < -\varepsilon$ the activity flag of the ith spot is raised; such spots are denoted by ξ_i^- and are referred to as generating spots. In the other case, if the spot's history flag is up, the newborn particle which is linked to ξ_i is freed; then, both flags of ξ_i are lowered.
- For each generating spot whose history flag is down, proceed as follows: Create a new particle which corresponds to the amount of vorticity which has been shed by the generating spot during the time interval (t_k, t_{k+1}); the position, strength, and volume of the particle in question are given by

$$\mathbf{x}_j(t_{k+1}) = \xi_i^- + \frac{\delta t}{2}\mathbf{u}(\xi_i^-, t_k). \tag{4.2}$$

$$a_j = -\delta l_i\, \delta t\, \mathbf{u}(\xi_i^-, t_k)\cdot\mathbf{n}_0(\xi_i^-) \tag{4.3}$$

$$\omega_j(t_{k+1}) = \omega_1^k(\xi_i^-, t_k). \tag{4.4}$$

If the distance $|\mathbf{x}_i(t_{k+1}) - \xi_i^-|$ is less than $\delta l_i/2$, the spot's history flag is raised and the particle which has been generated is given a status of a newborn particle that is still linked to ξ_i^-. If the distance from the particle to the generating spot is greater than $\delta l_i/2$, the spot's history flag is kept down and the newborn particle is freed.

- Let ξ_i^- be a generating spot whose history flag is up. Consider $(\mathbf{x}_j(t_k), \omega_j(t_k), a_j)$ the old position, strength, and volume of the newborn particle that is still linked to the spot. Since the newborn particle belongs to the set J_k of the particles which already exist, its new position and strength are calculated according to the algorithm which is described in the next subsection; let $(\mathbf{x}_j(t_{k+1}), \omega_j(t_{k+1}))$ be the new position and strength of the particle in question. During the time interval (t_k, t_{k+1}), the generating spot has shed a new particle $(\mathbf{x}_j'(t_{k+1}), \omega_j'(t_{k+1}), a_j')$ according to Eqs. (4.2), (4.3), and (4.4); then, the old particle and the new one are lumped into one particle according to the following scheme:

$$a_j \leftarrow a_j + a_j', \tag{4.5}$$

$$\mathbf{x}_j(t_{k+1}) \leftarrow \frac{a_j\mathbf{x}_j(t_{k+1}) + a_j'\mathbf{x}_j'(t_{k+1})}{a_j + a_j'} \tag{4.6}$$

$$\omega_j(t_{k+1}) \leftarrow \frac{a_j\omega_j(t_{k+1}) + a_j'\omega_j'(t_{k+1})}{a_j + a_j'}. \tag{4.7}$$

If the distance between the resulting particle and the generating spot is less than $\delta l_i/2$, the spot keeps its history flag up and the particle keeps its status of newborn particle that is still linked to its generating spot. Otherwise, the particle is freed and the spot's history flag is lowered.

4.2. *Convection and Diffusion of Particles of J_k*

In order to calculate the new locations of the particles of the set J_k, a numerical approximation of the velocity field (2.4) is needed. For this purpose a particle approximation of the inner vorticity field ω_1^k is sought. Recall that in the inner domain Ω_1, the vorticity field is known at the nodes $(\mathbf{x}_{ij})$ of a finite differences lattice; denote by $(\omega_1^k(\mathbf{x}_{ij}))$ the finite differences approximation of the vorticity. Consider a quadrature formula in Ω_1 so that for a certain class of smooth functions ϕ we have

$$\int_{\Omega_1} \phi(\mathbf{x})\, d\mathbf{x} = \sum_{l \in L} \beta_l \phi(\gamma_l), \tag{4.8}$$

where $(\beta_l)_{l \in L}$ are positive weights and $(\gamma_l)_{l \in L}$ are quadrature points. Interpolate the finite differences approximation of ω_1^k on the quadrature points $(\xi_l)_{l \in L}$; let $(\omega_1^k(\gamma_l))_{l \in L}$ be the result of such an interpolation. A particle approximation of ω_1^k is then given by

$$\Pi^h \omega_1^k = \sum_{l \in L} \beta_l \omega_1^k(\gamma_l)\, \delta(\mathbf{x} - \gamma_l). \tag{4.9}$$

Note that the position and strength of the particles in domain Ω_1 are time independent. For the sake of legibility let us set $\Theta_l = \beta_l \omega_1^k(\gamma_l)$, $\Theta_j = a_j \chi_{j^k}(t)\, \omega_j^k(t)$, $\mathbf{x}_l = \gamma_l$, and $\mathbf{x}_j = \mathbf{x}_j(t)$; then a classical approximation of the velocity field is given by

$$\mathbf{u}(\mathbf{x}) \approx \mathbf{u}_\infty(t_{k+1}) + \sum_{i \in J_k \cup L} \Theta_i \nabla G_\varepsilon(\mathbf{x} - \mathbf{x}_i) \times \mathbf{k}, \tag{4.10}$$

where ∇G_ε is a regularized approximation of the kernel ∇G (e.g., see Raviart [13] for details on the regularization technique). Indeed the sum appearing in (4.10) is calculated by means of a fast algorithm of the type that has been developed by Greengard and Rokhlin [5].

The new locations of the particles of J_k are obtained by solving the differential systems (3.16) by means of the following scheme which is second-order accurate in time:

$$\mathbf{x}_j(t_{k+1}) = \mathbf{x}_j(t_k) + \frac{3\delta t}{2} \mathbf{u}(x_j(t_k), t_k) - \frac{\delta t}{2} \mathbf{u}(x_j(t_{k-1}), t_{k-1}). \tag{4.11}$$

The viscous diffusion of vorticity is taken into account through the following classical conservative scheme (e.g., Raviart [13]):

$$\omega_i(t_{k+1}) = \omega_i(t_k) + \sum_{j \in J_k} \frac{a_j}{4\pi\nu\, \delta t} [\omega_j(t_k) - \omega_i(t_k)] \times \exp\left[-\frac{|\mathbf{x}_j(t_k) - \mathbf{x}_i(t_k)|^2}{4\nu\, \delta t} \right]. \tag{4.12}$$

4.3. *Absorption of Outgoing Particles*

We now describe how outgoing particles are smoothly absorbed. Once a newborn particle is freed from its generating spot, the particle is represented by a disk $d_j(t_k)$ whose surface is equal to the particle's volume a_j and whose center is located at $\mathbf{x}_j(t_k)$. Consider such a free particle of index $j \in J_k$. During the time interval (t_k, t_{k+1}), this particle is convected and diffused according to the algorithm presented in Section 4.2; let $(\mathbf{x}_j(t_{k+1}), \omega_j(t_{k+1}))$ be its new position and strength, and let a_j be its volume. Then, there are three possible cases:

- If $d_j(t_{k+1})$ is out of Ω_0, the particle is no longer taken into account in further computations; that is to say, the particle is excluded from J_{k+1}.
- If $d_j(t_{k+1})$ is in Ω_0, the particle is retained and stored in J_{k+1}.
- If $d_j(t_{k+1}) \cap \Gamma_1$ is not empty, the particle's volume is replaced by the volume of the domain $d_j(t_{k+1}) \cap \Omega_0$, and the new particle's position is defined as being the geometric barycenter of $d_j(t_{k+1}) \cap \Omega_0$. In algorithmic transcription we have

$$\omega_j(t_{k+1}) \leftarrow \omega_j(t_{k+1}), \tag{4.13}$$

$$a_j(t_{k+1}) \leftarrow \text{volume}[d_j(t_{k+1}) \cap \Omega_0], \tag{4.14}$$

$$\mathbf{x}_j(t_{k+1}) \leftarrow \text{barycenter}[d_j(t_{k+1}) \cap \Omega_0]. \tag{4.15}$$

Note that the above algorithm allows the particles to leave Ω_0 smoothly as it is recommended by Eq. (3.15).

5. THE FINITE DIFFERENCES APPROXIMATION IN Ω_1

5.1. *Considerations on Stability*

Since different numerical techniques are used in Ω_0 and Ω_1, there is no reason for the stability conditions to be the same in each subdomain. In other words, different time steps may be used in Ω_0 and Ω_1. Indeed, it has been found that, due to time linearization in Ω_1, the CFL condition is more restrictive in the inner domain than in the outer one. In practice, stability is achieved if the time step in Ω_1 is five times smaller than that in Ω_0. As a consequence, the domain decomposition algorithm which has been presented in

Section 2.4 must be slightly modified so that the stability criteria are met. For this purpose the time interval (t_k, t_{k+1}) is divided into M equal subintervals and an inner time step $\delta t' = \delta t/M$ is defined. Denoting by $\psi_1^{k,m}$ and $\omega_1^{k,m}$ approximations of ψ_1 and ω_1 on the time interval $(t_k + (m-1)\,\delta t',\ t_k + m\,\delta t')$, where $1 \leqslant m \leqslant M$, the system of partial differential equations to be solved in Ω_1, reads

$$\frac{\partial \omega_1^{k,m}}{\partial t} + \nabla \cdot (\omega_1^{k,m} \nabla \times (\psi_1^{k,m} \mathbf{k})) = \nu \nabla^2 \omega_1^{k,m}$$
$$\nabla^2 \psi_1^{k,m} = -\omega_1^{k,m}. \tag{5.1}$$

This problem is supplemented with boundary conditions on B_1 and transmission conditions on Γ_1 as follows.

5.2. *Approximation of the Boundary Conditions on B_1*

Assume for the sake of simplicity that the domain Ω_1 is mapped to the unit square $[0, 1] \times [0, 1]$ by a conformal mapping. Denote by (x, y) the coordinate system in the unit square's frame. Let $\omega_1^{k,m}(x, 0)$ and $\psi_1^{k,m}(x, 0)$ be the boundary values of $\omega_1^{k,m}$ and $\psi_1^{k,m}$ on the obstacle, and denote by δy the discretization step in the y direction. The formulae used for approximating the boundary values of $\omega_1^{k,m}$ and $\psi_1^{k,m}$ on the obstacle are

$$\psi_1^{k,m}(x, 0) = \frac{1}{\text{meas}(\Gamma_1)} \left[\int_{B_1} \psi_1^{k,m-1}(\xi, \delta y)\, dl(\xi) + \frac{\delta y^2}{2} \int_{B_1} \omega_1^{k,m-1}(\xi, \delta y)\, dl(\xi) \right] \tag{5.2}$$

$$\omega_1^{k,m}(x, 0) = \frac{2}{\delta y^2} [\psi_1^{k,m}(x, 0) - \psi_1^{k,m-1}(x, \delta y)], \tag{5.3}$$

where $\psi_1^{k,0} = \psi_1^{k-1,M}$, $\omega_1^{k,0} = \omega_1^{k-1,M}$, and $k \geqslant 1$.

The two formulae above are first-order accurate both in space and time. They have been found to be stable for reasonable time steps. Other choices are possible (e.g., see Gresho [6] for a review on this issue).

5.3. *Approximation of Transmission Conditions*

The transmission condition on $\omega_1^{k,m}$ is readily derived from (2.11) and is given by

$$\omega_1^{k,m}(\mathbf{x}) = \omega_0^k(\mathbf{x} - m\,\delta t'\,\mathbf{u}^k), \tag{5.4}$$

if $\mathbf{u}^k \cdot \mathbf{n}_1 < 0$, and

$$\omega_1^{k,m}(\mathbf{x}) = \omega_1^{k,m-1}(\mathbf{x} - \delta t'\,\mathbf{u}^k), \tag{5.5}$$

if $\mathbf{u}^k \cdot \mathbf{n}_1 > 0$, where $\mathbf{n}_1$ denotes the unit outward normal of domain Ω_1.

The approximation of $\psi_1^{k,m}$ on Γ_1 requires that a surface integral on the domain $\Omega_0 \cup \Omega_1$ (see Eq. (2.8)) is approximated. Note that the quadrature technique which has been developed for approximating the Biot and Savart's integrals of the velocity field can be applied to the present problem. That is to say, a suitable approximation of $\psi_1^{k,m}$ on Γ_1 is given by

$$\psi_1^{k,m}(\mathbf{x}) \approx u_\infty(t_{k+1})\, \mathbf{x} \cdot \mathbf{j} - \sum_{i \in J_k \cup L} \Theta_i G_\varepsilon(\mathbf{x} - \mathbf{x}_i) - \psi_{B_1}^{k+1} \int_{B_1} \frac{\partial G(\mathbf{y} - \mathbf{x})}{\partial n}\, dl, \tag{5.6}$$

where G_ε is a regularized approximation of G, $(\Theta_i)_{i \in J_k}$ is the particle approximation of ω_0^k, and $(\Theta_i)_{i \in L}$ is the particle approximation of $\omega_1^{k,m-1}$.

The remaining line integral on B_1 in (5.6) is approximated by means of any standard one-dimensional quadrature.

5.4. *Numerical Implementation*

After a suitable linearization in time, the problem above (i.e., (5.1) + ((5.2), (5.3), (5.4), (5.5), (5.6))) is solved by means of a finite differences method which is second-order accurate in space and first-order accurate in time. The method has been devised by Daube and Loc [4]. The flow domain Ω_1 is mapped to the unit square, and the convection-diffusion equation for the vorticity is solved by means of an alternate direction implicit (ADI) procedure. Centered differences are used for the discretization of the spatial derivatives. The Poisson equation for the stream function is also solved by an ADI scheme (see [4] for other details). Note that other numerical approximations which are suitable for solving elliptic problems can be used, for in domain Ω_1 the elliptic nature of the flow is dominant (e.g., see Gresho [6] for a review on the numerical techniques that can be used).

6. NUMERICAL RESULTS AND COMPARISONS

In order to illustrate the present method, comparisons with well-documented experimental and numerical results have been performed. In a first series of tests, we have compared the results of the domain decomposition method with that of the finite differences method when it is applied to the whole domain Ω. In Fig. 2 are reported the streamline patterns of the flow about an almost-impulsively started cylinder. The length scale is the cylinder radius, r, and the velocity scale is the ultimate velocity at infinity, V_∞. The results from the domain decomposition method are plotted on the right of the figure and that of the finite differences method are plotted on the left. The ultimate Reynolds number, $\text{Re} = 2V_\infty r/\nu$, is equal to 3000. The interface Γ_1 is a circle the radius of which is twice that of the cylinder. The

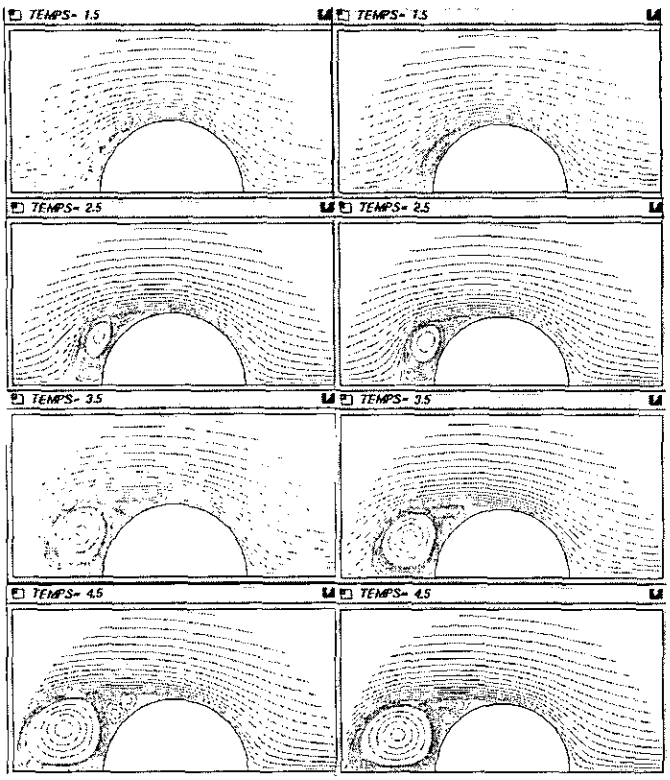

FIG. 2. Domain decomposition method (right) versus finite difference method on the whole fluid domain (left). Streamline patterns about an almost-impulsively-started cylinder.

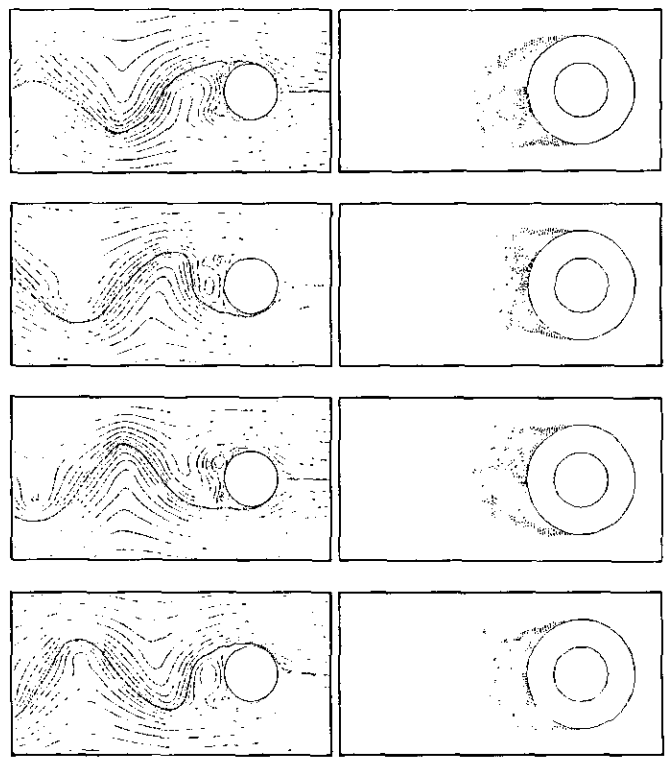

FIG. 4. Karmann street behind a cylinder, Re = 3000. Streamline patterns (left), particles (right). From top to bottom: $t = 26$, 28, 30, and 32.

good agreement between the two approaches is an indication that the present domain decomposition method is consistent.

In a second series of tests, the numerical results have been compared with experimental data (see Loc and Brouard [11] for details on the experiment). In Fig. 3 are reported comparisons between experimental (symbols) and numerical results (solid lines) for the radial velocity on the symmetry axis behind the above mentioned cylinder. The qualitative agreement between the results of the decomposition method and the experimental data is quite good on the whole range of time. Note that a slight discontinuity in the slope of the numerical results is noticeable at $x/r = 2$. This radius is that of the interface Γ_1 between the two domains.

In order to illustrate the fact that the present method naturally puts discretization points in the flow regions which are of interest, the calculation of the flow about the above-mentioned cylinder has been slightly perturbed so that a Karmann street spreads out. The streamline patterns and the particles which have been shed between $t = 26$ and $t = 32$ are plotted in Fig. 4; the time scale is r/V_∞. The streamline patterns are plotted on the left and the particles are plotted on the right. Note that the particles that are shed in Ω_0 closely trace the vorticity in the cylinder's wake; hence, no computational effort is dedicated to the flow region where the vorticity is zero. The inner domain Ω_1 (shown on the right-hand side figures) is a ring the width of which is equal to one cylinder radius. The Strouhal number

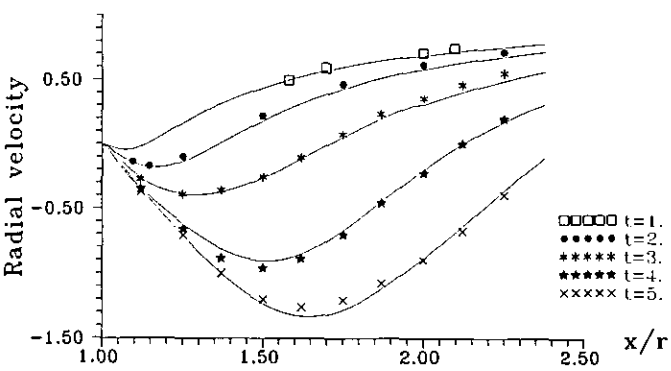

FIG. 3. Comparison between numerical (solid lines) and experimental (symbols) radial velocity profile on the symmetry axis behind the cylinder for Re = 3000.

FIG. 5. Streaklines about a cylinder at $t = 50$, Re = 3000.

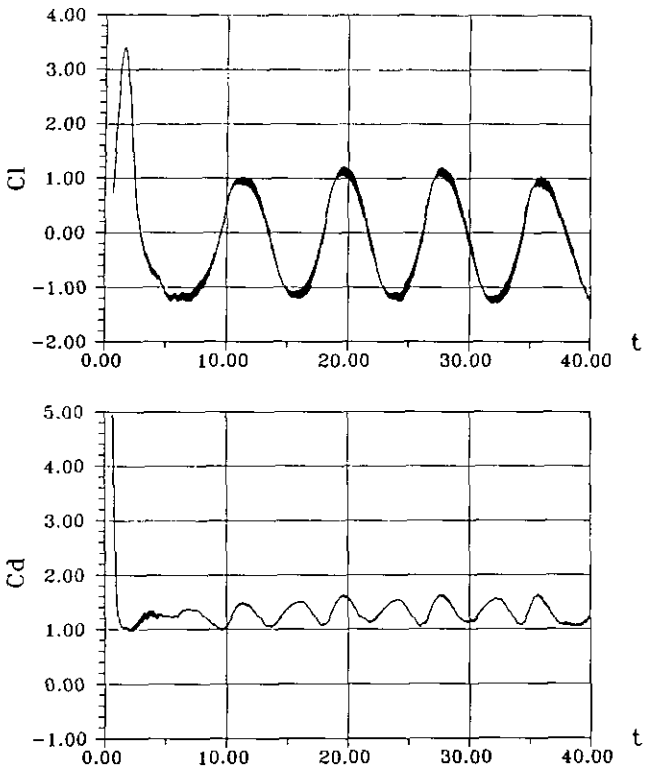

FIG. 6. History of lift (top) and drag (bottom) coefficients on the cylinder $0 < t \leqslant 40$, Re = 3000.

associated with the Karmann street is approximately equal to 0.24. The experimental Strouhal number is about 0.21.

Figure 5 shows the streaklines about the impulsively started cylinder at time $t = 50$. The Reynolds number is equal to 3000 and the finite differences domain is still reduced to a ring, the width of which is set to one cylinder radius. The Karmann street is clearly shown. This figure illustrates the fact that the present technique is suitable for convecting coherent structures on large distances without introducing significant numerical diffusion. Moreover, this technique does not require a transparent outflow boundary condition on a distant artificial boundary.

In order to illustrate suitability of the present approach for computing fluid–structure interactions we have plotted the drag and lift time history of the above cylinder for $0 < t \leqslant 40$ in Fig. 6. Note that the frequency of the drag oscillations is twice that of the lift oscillations, as expected. The thickness of the lift coefficient curve is due to high frequency numerical oscillations of small amplitude.

7. CONCLUSION

In this paper we have presented a domain decomposition method for simulating external incompressible viscous flows. One original aspect of the present work consists in coupling of two different numerical techniques within the framework of a domain decomposition method. Another original contribution consists in the development of a vortex method which is suitable for approximating first-order linear hyperbolic problems supplemented with Dirichlet data.

For the sake of simplicity the flow domain has been assumed to be one-connected; indeed, the present method can be readily extended to the p-connected case with p moving obstacles. Calculations with two and three moving obstacles have been carried out and are reported in [7]. The present approach is believed to be particularly adapted to the multiple-moving-obstacle problem.

In order to increase the efficiency of the method some technical improvements may be implemented. For example, the ADI scheme that is used for solving the Poisson problem of the inner problem may be replaced by a multigrid or a FFT algorithm. Some other formulations of the Navier–Stokes equations can be used in Ω_1; some tests on the velocity-pressure formulation are under way. The present approach is also being extended to problems involving heat transfer.

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